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Extension of the Ablowitz–Ladik Method to the Derivative Nonlinear Schrödinger Equation

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The derivative nonlinear Schrodinger equation is solved by application of the Ablowitz–Ladik scheme to an equivalent equation The variations of the results due to modifications in the spatial grid size and time step are analyzed The scheme maintains the main properties of the original equation and allows the use of rather large time steps © 1988 Academic Press, Inc

I. INTRODUCTION

In [1] Ablowitz and Ladik generalized the theory of the inverse scattering transform (IST) to cover nonlinear partial difference equations. This generalization is particularly useful for developing numerical schemes of nonlinear evolution equations which maintain the main properties of the original equations. The procedure is as follows. One proposes a discrete version of the standard Ablowitz–Kaup–Newell–Segur (AKNS) eigenvalue problem [2] of the form [1, 3]

$$\begin{aligned} v_{1n+1}^m &= z v_{1n}^m + Q_n^m v_{2n}^m + S_n^m v_{2n+1}^m \\ v_{2n+1}^m &= \frac{1}{z} v_{2n}^m + R_n^m v_{1n}^m + T_n^m v_{1n+1}^m, \end{aligned} \tag{1.1}$$

where n refers to the spatial grid point and m to the time level, z is the eigenvalue,

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v_1 and v_2 are the corresponding eigenfunctions and R_n^m, Q_n^m, S_n^m, and T_n^m are the potentials. Then, the eigenfunctions are advanced in time according to

$$\begin{aligned} v_{1n}^{m+1} - v_{1n}^m &= A_n^m(z)\, v_{1n}^m + B_n^m(z)\, v_{2n}^m \\ v_{2n}^{m+1} - v_{2n}^m &= C_n^m(z)\, v_{1n}^m + D_n^m(z)\, v_{2n}^m \end{aligned} \tag{1.2}$$

and one forces the consistency condition

$$\Delta^m(E_n v_{in}^m) = E_n(\Delta^m v_{in}^m) \qquad i = 1, 2, \tag{1.3a}$$

where

$$E_n(v_{in}^m) = v_{in+1}^m; \qquad \Delta^m v_{in} = v_{in}^{m+1} - v_{in}^m \tag{1.3b}$$

are the space and time displacement operators, respectively. Finally one expands A_n^m, B_n^m, C_n^m, and D_n^m in powers of z and $1/z$, imposes (1.3) using (1.1) and (1.2), and equates terms with equal powers of z. This procedure gives relations between the coefficients of the expansions of A_n^m, B_n^m, C_n^m, D_n^m and the potentials, and it also gives the evolution equation that the potentials must satisfy in order that (1.3) is valid. That is, the evolution equations are the integrability condition of the eigenvalue problem. It is clear that the menthod parallels the standard procedure of Ablowitz *et al* [2]: if one chooses different expansions for A_n^m, B_n^m, C_n^m, and D_n^m and different relations between the potentials one obtains different evolution equations. In this way the NLS, KdV, MKdV, and S–G equations appeared in Ref. [2] as the integrability condition of a particular continuous version of (1.1) and (1.2). It is then clear that this method is suitable to obtain the difference scheme for some nonlinear equations of the above mentioned class. In [3] Taha and Ablowitz gave the discrete version of the NLS, KdV, and MKdV equations which can be used for their numerical integration [4, 5]. The main differences between the equations obtained in this way and the ones one would obtain by writing the original nonlinear differential equations directly in finite difference form, appear in the expression of the nonlinear terms. It is then crucial that these terms be expressed in a form which maintains the main properties of the equations which are completely integrable by the IST method.

However, we cannot apply the above procedure to the derivative nonlinear Schrödinger equation (DNLS), which describes the evolution of nonlinear Alfvén waves,

$$iq_t - i(q^2 r)_x + q_{xx} = 0; \qquad r = -\mathrm{NS}\, q^* \tag{1.4}$$

where NS is a sign ($\mathrm{NS} = \pm 1$), since Eq. (1.4) is the integrability condition of

$$\begin{aligned} v_{1x} + i\xi^2 v_1 &= \xi q v_2, \\ v_{2x} - i\xi^2 v_2 &= \xi r v_1, \end{aligned} \tag{1.5}$$

with

$$\begin{aligned} v_{1t} &= Av_1 + Bv_2, \\ v_{2t} &= Cv_1 + Dv_2, \\ D &= -A \end{aligned} \tag{1.6}$$

and (1.5) is different from the standard AKNS eigenvalue problem. A, B, and C are functions of q and r, whose detailed form is not relevant to this discussion (see Eq. (1.11) below for an equivalent useful expression). Furthermore, when one expands A, B, C, and D in powers of ξ and equates terms with equal powers, the evolution equation (1.4) is obtained for the ξ^1 term, while, for the standard AKNS case it is always obtained for the ξ^0 one. This fact, which is due to the factor ξ in front of the "potentials" q and r in (1.5), introduces some asymmetry in the problem and it is not clear how one can generalize the discrete version (1.1) to this case. However, there is a transformation of the original eigenfunctions v_1 and v_2 which leads (1.5) to the standard AKNS problem. This transformation is given by [6]

$$\begin{aligned} v_1' &= v_1 \exp(-i\mu) \\ v_2' &= \xi v_2 \exp(i\mu) - \frac{i}{2} v_1 r \exp(i\mu), \end{aligned} \tag{1.7}$$

where $\mu = \int_{-\infty}^{x} rq/2\, dx$, and it has been used in [6] for solving the DNLS by the IST. In this way, the eigenvalue problem (1.5) may be written as the standard one

$$\begin{aligned} v_{1x}' + i\lambda v_1' &= Qv_2', \\ v_{2x}' - i\lambda v_2' &= Rv_1', \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} Q &= q \exp(-2i\mu) \\ R &= -\frac{i}{2}\left(r_x + i\frac{qr^2}{2}\right) \exp(2i\mu), \\ \lambda &= \xi^2. \end{aligned} \tag{1.9}$$

The evolution equations for R and Q are

$$\begin{aligned} iQ_t - 2Q^2R + Q_{xx} &= 0 \\ iR_t + 2R^2Q - R_{xx} &= 0 \end{aligned} \tag{1.10}$$

and may be obtained as the integrability condition of (1.8) together with the

corresponding evolution equations for v_1' and v_2' (which are similar to the ones given in (1.6)), expanding A', B', C', and D' as

$$\begin{aligned} A' &= -iQR - 2i\lambda^2 \\ B' &= iQ_x + 2Q\lambda \\ C' &= -iR_x + 2R\lambda \\ D' &= A'. \end{aligned} \tag{1.11}$$

This is the same expansion that must be performed in order to obtain the NLS equation. It is clear from (1.10) that the evolution equations for Q and R are similar to the NLS system. However, the difference is the relation between R and Q, which is $R = \mp Q^*$ for the NLS and

$$R = -\frac{1}{4}|Q|^2 Q^* + \mathrm{NS}\,\frac{i}{2} Q_x^* \tag{1.12}$$

for the present case.

II. Extension of the NLS Results to the Transformed Problem

The similarity between the equations for Q and R and the NLS suggests a simple way to obtain their discrete version. We follow the steps described in the Introduction, taking $T_n^m = S_n^m = 0$, and choose the expansions

$$\begin{aligned} A_n^m &= A_n^{(-2)} z^{-2} + A_n^{(0)} + A_n^{(2)} z^2 \\ B_n^m &= B_n^{(-1)} z^{-1} + B_n^{(1)} z^1 \\ C_n^m &= C_n^{(-1)} z^{-1} + C_n^{(1)} z^1 \\ D_n^m &= D_n^{(-2)} z^{-2} + D_n^{(0)} + D_n^{(2)} z^2 \end{aligned} \tag{2.1}$$

as in the case of the NLS equation [3]. We then impose the consistency condition (1.3) and obtain a discrete version of the evolution equation for Q. If we require that in the linear limit, as it is done in [3]

$$\frac{i\Delta^m Q_n^m}{\Delta t} = -\frac{1}{2(\Delta x)^2}\left(Q_{n+1}^m - 2Q_n^m + Q_{n-1}^m + Q_{n+1}^{m+1} - 2Q_n^{m+1} + Q_{n-1}^{m+1}\right) \tag{2.2}$$

then the evolution equation reads as

$$\frac{i\Delta^m Q_n^m}{\Delta t} = -\frac{1}{2(\Delta x)^2}(Q_{n+1}^m - 2Q_n^m + P_{n-1}Q_{n-1}^m + P_n Q_{n+1}^{m+1} - 2Q_n^{m+1} + Q_{n-1}^{m+1})$$
$$+\tfrac{1}{2}\Bigg(P_n R_n^m Q_n^m Q_{n+1}^{m+1} + P_{n-1}R_n^{m+1}Q_{n-1}^m Q_n^{m+1}$$
$$+\tfrac{1}{2}(Q_n^m Q_{n+1}^m R_n^m + Q_{n-1}^{m+1}Q_n^{m+1}R_n^{m+1} + Q_{n-1}^m Q_n^{m+1}R_n^m$$
$$+Q_n^m \sum_{k=-N}^{n} \Delta^m \tilde{S}_k^m - Q_n^{m+1}\sum_{k=-N}^{n-1}\Delta^m \tilde{T}_k^m\Bigg)\Bigg), \tag{2.3}$$

where

$$\tilde{S}_k^m = Q_{k+1}^m R_k^m + Q_k^m R_{k-1}^m; \qquad \tilde{T}_k^m = Q_{k-1}^m R_k^m + Q_k^m R_{k+1}^m$$
$$P_n = \prod_{k=-\infty}^{n}[(1 - R_k^{m+1}Q_k^{m+1}(\Delta x)^2)/(1 - R_k^m Q_k^m(\Delta x)^2)],$$

which is similar to Eq. (2.8) of Ref. [3], but where R_n^m has not been replaced by Q_n^{m*}. The problem in this case is that the relation between R and Q is nonlinear and the method gives no answer on how to write it. Nevertheless, since in the discrete version of the AKNS the qv term is converted into $Q_n^m v_n^m$, we choose

$$R_n^m = \frac{i}{2}\,\mathrm{NS}\cdot\left(\frac{Q_{n+1}^{m*} - Q_{n-1}^{m*}}{2\Delta x}\right) - \frac{1}{4}Q_n^m(Q_n^{m*})^2. \tag{2.4}$$

III. Numerical Algorithm and Results

We have used Eq. (2.3) in order to numerically solve the equation for Q, for both signs in the nonlinear term (NS = 1 and NS = −1) of the original DNLS equation. For simplicity, we employed a local scheme (which is equivalent to set $P_n = 1$, $\tilde{S}_n^m = \tilde{T}_n^m = 0$ in (2.3)). We have chosen periodic boundary conditions with $2N+1$ spatial grid points ($n \leqslant N$). We have used a numerical algorithm which is explained in Section 2 of Ref. [4]. The only differences with the formulae given in [4] are the expression for R and the sign of $\varepsilon = -2i((\Delta x)^2/\Delta t)$. This last difference is due to the fact that we solve an equation with a different sign in the dispersion term than the one that is solved in [4]. The algorithm chosen involves two nested iterations. The outer one is related to the fact that the expression (2.3) is highly implicit and contains nonlinear terms in which Q_n values at the new time level $(m+1)$ appear. Therefore, we first take $Q_n^{m+1} = Q_n^m$ in those nonlinear terms and solve a linear system at the new time level. The inner iteration, is related to the way this linear system is solved, since the corresponding matrix is not inverted, but the values Q_n^{m+1} at different spatial grid points are obtained by an iterative procedure. For both iterations we use, as in Ref. [4], the Crank–Nicholson back and forth sweep

method for the heat equation. If we label by j the outer iteration number, the whole procedure is stopped when

$$|Q_n^{m+1,j} - Q_n^{m+1,j+1}|/|Q_n^{m+1,j}| < \text{tolerance}$$

and the value $Q_n^{m+1,j+1}$ is taken as the approximate solution at time $t = (m+1)\,\Delta t$ and position $x = n\,\Delta x$. The algorithm requires a small $(\Delta x)^2/\Delta t$ ratio and allows the use of a rather large time step. This proved to be useful for the determination of the long-time behaviour of the solution, since it can be obtained with less computational effort than with other methods, as the one used in Ref. [7]. We have considered 500 x-grid points and $\Delta t \geqslant 0.2$.

We have implemented the code for the initial conditions $Q(x, 0) = q \exp(i\,\mathrm{NS} \int_{-\infty}^{x} |q(x', 0)|^2\,dx)$ with $q(x, 0)$ given by:

1. One soliton solution

$$\begin{aligned} q(x, 0) &= u \exp(i\varphi) \\ u^2 &= 8\Delta^2 \sin^2\gamma/[\mathrm{ch}(4\Delta^2 \sin\gamma x) + \cos\gamma] \\ \varphi &= -2\,\mathrm{NS}\,\Delta^2 \cos\gamma x - \frac{3}{4}\,\mathrm{NS} \int_{-\infty}^{x} u^2\,dx \end{aligned} \tag{3.1}$$

for $0 < \gamma < \pi$, which represents a soliton of velocity $v = -4\,\mathrm{NS}\,\Delta^2 \cos\gamma$ and

ii. Modulated plane wave

$$q(x, 0) = A_0 \exp\left(\frac{2\pi i x}{\lambda}\right) \exp\left(-\frac{x^2}{L^2}\right), \tag{3.2}$$

where λ is the wavelength, A_0 the amplitude, and L the modulation scalelength of the Gaussian envelope.

The code preserves the solitons' identity as it may be seen in Fig. 1 which corresponds to solitons with $v = 0.5$ and different amplitude. It also maintains the symmetries of the equation. As we have already said, we have solved the equation for $\mathrm{NS} = 1$ and $\mathrm{NS} = -1$. However, it is clear from (1.4) that to change NS from 1 to -1 is equivalent to making the change $x \to -x$. If we also make this change in the initial condition, we obtain

$$q_+(x, t) = q_-(-x, t), \tag{3.3}$$

where q_+ is the solution of the DNLS with positive nonlinearity ($\mathrm{NS} = 1$) and q_- the corresponding one to negative nonlinearity ($\mathrm{NS} = -1$). This feature is illustrated in Fig. 2, where the evolution of the envelope ($|q| = |Q|$) from the initial condition (3.2) is plotted. It is clear that, for this initial condition, the changes $x \to -x$ and $\lambda \to -\lambda$ are equivalent. Fig. 2a corresponds to $\mathrm{NS} = 1$ and $\lambda = 5$ and Fig. 2b to $\mathrm{NS} = -1$ and $\lambda = -5$. The evolution of the initially modulated plane

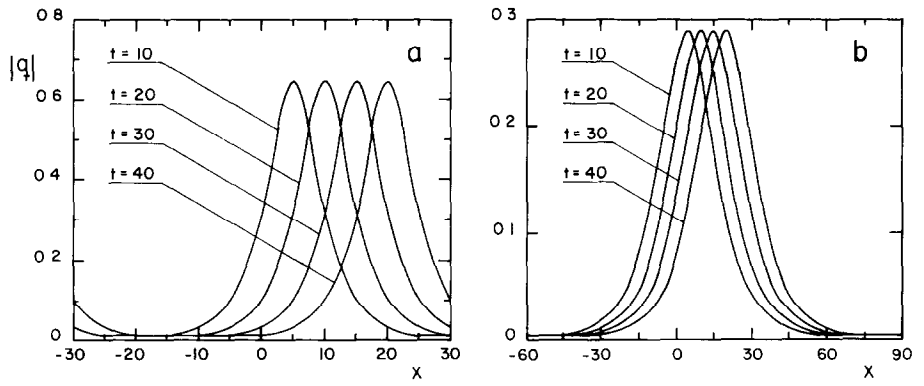

FIG 1 Propagation of an envelope soliton of the form (3 1) (a) corresponds to $\gamma = \pi/4$, $\Delta^2 = \sqrt{2}/8$, NS $= -1$ The velocity of propagation is $v = 4\Delta^2 \cos\gamma = 0\,5$ $\Delta t = 0\,2$, $\Delta x = 0\,12$, 2N $= 500$, (b) corresponds to $\gamma = \pi/8$, $\Delta^2 = 0\,1353$, NS $= -1$ The velocity of propagation is $v = 4\Delta^2 \cos\gamma = 0\,5$, $\Delta t = 0\,2$, $\Delta x = 0\,36$, 2N $= 500$

waves also agrees with the prediction of the IST theory: the pulse decays into a soliton train plus a dispersive residue (radiation). We have obtained that, for this initial condition the number of solitons depends on the constant $C1$,

$$C1 = \int_{-\infty}^{\infty} RQ\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \left(-iqr_x + \frac{q^2 r^2}{2} \right) dx. \tag{3.4}$$

If $C1$ is positive, there is no soliton and if it is negative, the number of solitons is an increasing function of its absolute value. Finally the code also keeps the values of

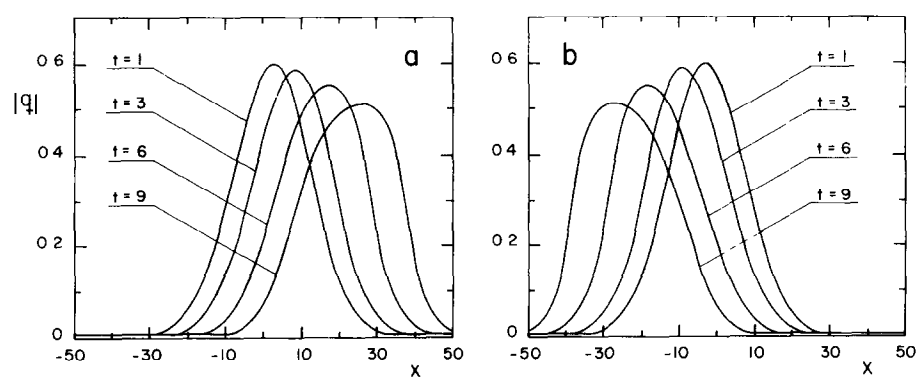

FIG 2 Evolution of an initial pulse of the form (3 2) (modulated plane wave) with $A_0 = 0\,6$ and $L = 14$ (a) corresponds to positive nonlinearity (NS $= 1$) and $\lambda = 5$, (b) corresponds to negative nonlinearity (NS $= -1$) and $\lambda = -5$ It is clear that $|q_{(a)}(x, t)| = |q_{(b)}(x, t)|$ and so, the changes $x \to -x$ and $\lambda \to -\lambda$ are equivalent In both cases is $\Delta t = 0.2$, $\Delta x = 0\,2$, 2N $= 500$

the constants of motion of the DNLS equation. When the conservation of the two first constants of motion $C0$,

$$C0 = \int_{-\infty}^{\infty} |q|^2 \, dx = \int_{-\infty}^{\infty} |Q|^2 \, dx \tag{3.5}$$

and $C1$ is tested, it may be seen that the relative differences between their values at different instants are less than 5%. The conservation is better for solitons than for modulated plane waves. For those cases of an initial modulated plane wave with negative $C1$, the errors grow till the moment at which the pulse achieves its maximum peak value and minimum length scale, that is, just before the formation of the soliton train. At this moment the number of outer iterations (IT) to achieve a ratio $|(Q_n^{m+1,IT} - Q_n^{m+1,IT-1})/Q_n^{m+1,IT-1}|$ less than a given tolerance (0.01) is the highest one (in the examples we are analyzing it never exceeds the value of six iterations). Afterwards IT remains constant ($IT = 2, 3$) and the errors first decrease and finally increase monotonically. For positive $C1$ these errors grow monotonically from the beginning, while the number of iterations remains constant ($IT = 3$). Although IT is constant for the cases which give no soliton train, the errors are greater than for the other cases. Besides, for negative $C1$, it is evident that the characteristic time scales decrease with $C1$. This fact allows the use of rather big time steps without losing the properties of the solutions. The errors also depend on the time step, spatial grid size, and on the ratio $\Lambda = (\Delta x)^2/\Delta t$. If we let Δt invariant and lower Δx, then Λ is also reduced and the conservation of $C0$ and $C1$ is better However, the conservation of $C0$ is more affected than that of $C1$. If we let Δx invariant and increase Δt, the conservation of $C0$ is poorer. The errors of $C1$, instead, as Λ is lower, are hardly reduced for those steps with small IT, but then grow when many iterations are needed. We suppose that this different sensitivity to the variation of Δx and Δt is due to the fact that $C1$ is the first constant that the IST method affords. $C0$ may be prescribed independently, it does not depend on the scattering data [6]. As the numerical algorithm is closely related to the integration of the equation by the IST and only the Λ ratio enters in the computation of Q^{new} (Q^{old}), $C1$ is more affected by Λ variations and $C0$, by variations in Δx and Δt separately. Nevertheless, when the number of iterations IT is too large, $C1$ is better conserved for lower Δt values which give lower values of IT.

IV. Conclusions

We have applied the local Ablowitz–Ladik scheme to a nonlinear equation which is equivalent to the derivative nonlinear Schrödinger equation. The code maintains the main properties of the equation and allows the use of rather large time steps. These time steps may be chosen in such a way that the number of iterations needed to obtain Q^{new} (Q^{old}) is not too large. As the characteristic time scales vary during the integration, the code may be improved by including the possibility of a

modification of Δt during the integration. We have analyzed the conservation of the two first constants of motion $C0$ and $C1$. The conservation is better for solitons than for modulated plane waves and for the $NS = -1$ case better than for the $NS = 1$ one. The errors in the conservation, which do not exceed 5% of the constants values, depend on the time step and spatial grid size; while $C0$ is most affected by variations in Δx and Δt separately, $C1$ is affected by variations in the ratio $\Lambda = (\Delta x)^2/\Delta t$. We think better results could be achieved for lower Δx and Δt, and consequently, lower Λ values, but we did not try them because of computational limitations.

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